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On the solutions of the anisotropic Heisenberg equation

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Abstract. A method based on the discrete group of the inner symmetry of integrable systems is used to derive explicit formulae for soliton-like solutions of Heisenberg ferromagnets (with biaxial or uniaxial anisotropy). The solutions are given in terms of expressions which involve ratios of two determinants.

1. Introduction

In this paper we demonstrate how the use of the method of the discrete group of symmetries of integrable systems (auto-Backlund transformations) can be used to derive explicit expressions for the soliton solutions of such systems. As an example we study the Landau–Lifschitz equation (the equation for a Heisenberg ferromagnet with biaxial or uniaxial anisotropy [1]) as this equation is sufficiently complicated to demonstrate the usefulness of our approach and at the same time is very important because of its many applications in physics.

All the important properties of this equation are contained in the following chain of equations (in general, an unlimited one):

$$\frac{1}{\exp(\phi_{i+1}-\phi_i)+1}-\frac{1}{\exp(\phi_i-\phi_{i-1})+1}=\frac{(\ln(\phi_i'^2+(\alpha e^{2s}+\gamma+\alpha e^{-2s}))'}{2\phi_i'} \tag{1.1}$$

where ϕ_i are the unknown functions, s is an independent variable, $\phi_i'=d\phi_i/ds$, and α and γ are arbitrary parameters of the model (related to the moments of inertia of a non-axial-symmetric 'rigid body'). The transformation (1.1) describes the group of the discrete symmetries of the Landau–Lifschitz equation.

Let us add that this problem has been studied before. In fact, reference [2] considers this problem from the point of view of its Lax representation while [3] presents a discussion based on the Hamiltonian formalism.

2. Landau–Lifschitz equation

The Landau–Lifschitz equation arose out of the generalizations of the Heisenberg model of a homogeneous ferromagnet. In its original form [4] the equation describes the evolution

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of a unit vector field S ($S^2 = 1$), which is a function of one space variable (x) and time (t). This evolution is described by

$$\dot{S} = S \times S'' + S \times (\hat{J} S) \qquad \hat{J} = \mathrm{diag}(J_1, J_2, J_3) \tag{2.1}$$

where $\dot{}$ and $'$ denote the time and space derivatives, respectively. It is convenient [5] to perform a stereographic projection and so to introduce complex fields u and v:

$$u = \frac{S_1 + \mathrm{i}S_2}{1 + S_3} \qquad v = \frac{S_1 - \mathrm{i}S_2}{1 + S_3}. \tag{2.2}$$

Then (disregarding the condition of reality (i.e. that $v = u^\dagger$)) (2.1) becomes equivalent to the following set of two equations:

$$\begin{aligned} \dot{u} + u'' - 2v\frac{u'^2 + P(u)}{1 + uv} + \frac{1}{2}\frac{\partial}{\partial u}P(u) = 0 \\ -\dot{v} + v'' - 2u\frac{v'^2 + P(v)}{1 + uv} + \frac{1}{2}\frac{\partial}{\partial v}P(v) = 0 \end{aligned} \tag{2.3}$$

where $P(y) = \alpha y^4 + \gamma y^2 + \alpha$, $\dot{u}$ denotes $\mathrm{i}\partial u/\partial t$, and, as before, $'$ denotes the derivative with respect to x. Moreover, $\alpha = \frac{1}{4}(J_2 - J_1)$ and $\gamma = \frac{1}{2}(J_1 + J_2) - J_3$. In the case when the 'rigid body' is axially symmetric (i.e. when $J_1 = J_2$, or $J_2 = J_3$ or $J_1 = J_3$) we find that $\alpha = 0$ and $\gamma = \pm 2\alpha$, respectively.

The system of equations (2.3) is invariant with respect to the following discrete nonlinear transformation ($u \to U$, $v \to V$), where

$$U = \frac{1}{v} \qquad \frac{1}{1 + UV} - \frac{1}{1 + uv} = \frac{(\ln v)'' + \alpha(v^2 - v^{-2})}{[(\ln v)']^2 + \alpha v^2 + \gamma + \alpha v^{-2}}. \tag{2.4}$$

This can be verified by a direct computation, or checked by using, say, REDUCE. A similar transformation was found for other equations [6, 7] and was shown in [6] to have many far-reaching consequences; namely, it can be used to generate new solutions from the old ones.

Thus the transformation (2.4) plays the key role in our work. In fact, as in [6], we will use it in the following way. Instead of solving the original equations (2.3) we will consider (2.4) and treat it as an iterative procedure for generating from one set of functions u and v another one. Then having 'solved' this iterative procedure we will find that if we start from a given solution of (2.3) we will have many other solutions, among which we will be able to find the ones which satisfy the reality condition $u^\dagger = v$.

Thus, if we denote u and v as u_i and v_i, and U and V as u_{i+1} and v_{i+1}, respectively, we see that (2.4) becomes

$$u_{i+1} = \frac{1}{v_i} \qquad \frac{1}{1 + u_{i+1}v_{i+1}} - \frac{1}{1 + u_i v_i} = \frac{(\ln[((\ln v_i)')^2 + \alpha v_i^2 + \gamma + \alpha v_i^{-2}])'}{2(\ln v_i)'} \tag{2.5}$$

which, in what follows, we will call the Landau–Lifschitz lattice.

It is in this form that the invariance of the Heisenberg model (with biaxial or uniaxial anisotropy) was first considered in [6, 7].

Looking at (2.5) we observe that, in general, the chain of equations (2.5) is infinite except when

$$[(\ln v_i)']^2 + \alpha v_i^2 + \gamma + \alpha v_i^{-2} = 0\,. \tag{2.6}$$

In this singular case we cannot express u_{i+1} and v_{i+1} in terms of u_i and v_i.

However, the transformations (2.4) and (2.5) are invertible and so (2.5) can be rewritten as

$$\frac{1}{1+u_i v_i} - \frac{1}{1+u_{i+1}v_{i+1}} = \frac{(\ln[((\ln u_{i+1})')^2 + \alpha u_{i+1}^2 + \gamma + \alpha u_{i+1}^{-2}])'}{2u_{i+1}'}\,. \tag{2.7}$$

Thus, if equation (2.6) is also satisfied by some u_a then the Landau–Lifschitz chain is limited from both ends. In this case we have (2.5) together with the boundary conditions

$$[(\ln v_0)']^2 + \alpha v_0^2 + \gamma + \alpha v_0^{-2} = 0 \qquad [(\ln u_N)']^2 + \alpha u_N^2 + \gamma + \alpha u_N^{-2} = 0\,. \tag{2.8}$$

Similar equations have been studied before. In fact, reference [7] presents solutions of the corresponding discrete lattices for many integrable systems. The lattices which appear in [6] were all related (in a direct or an indirect form) to the Toda lattice. The chain described by (2.5) and (2.7) is more complicated and, as we will show, is related to the doubly periodic elliptic functions and contains the Toda lattice as a non-trivial limiting case.

3. Solution of the linear problem as the initial condition for the Landau–Lifschitz lattice

Our solution of the chain (2.5) will be presented in the next sections. Here we will discuss the constraints on the solutions of (2.3) which arise from the boundary conditions (2.8). Thus we want to find the initial functions u_0 and v_0 which satisfy (2.3). But let us observe that if we impose

$$u_0'^2 + P(u_0) = 0 \tag{3.1}$$

then $u_0'' + \frac{1}{2}(\partial/\partial u_0)P(u_0) = 0$ and so we see that the first equation in (2.3) is satisfied if $\dot{u}_0 = 0$. So u_0 is given by

$$\int^{u_0} \frac{\mathrm{d}y}{\sqrt{-P(y)}} = x + C \tag{3.2}$$

or we may find an expression for u_0 in terms of some elliptic functions.

Let us multiply the first equation in (2.3) by v_0, the second by u_0 and subtract. We find

$$-(u_0v_0)^{\cdot} + (v_0'u_0 - u_0'v_0)' - 2\alpha u_0v_0(v_0^2 - u_0^2) - 2(u_0v_0)'\frac{u_0v_0' - v_0u_0'}{1+u_0v_0} + 2\alpha\frac{(u_0^2v_0^2 - 1)(v_0^2 - u_0^2)}{1+u_0v_0} = 0\,. \tag{3.3}$$

Then we introduce $u_0 = \mathrm{e}^s$ and $Y = [1/(1+u_0v_0)] - \frac{1}{2}$ and observe that Y satisfies

$$-\dot{Y} + \left[s'\left(Y_s + \tfrac{1}{2} - 2Y^2\right)\right]' + 2\alpha\frac{v_0^2 - u_0^2}{(1+u_0v_0)^2} = 0 \tag{3.4}$$

where $Y_s = \partial Y/\partial s$. However, from (3.1) we see that $(s')^2 + \gamma + \alpha(e^{2s} + e^{-2s}) = 0$, $s'' + \alpha(e^{2s} - e^{-2s}) = 0$ and so we find that the equation for Y can be rewritten as

$$-\dot{Y} + (2PY^2)_s - (PY_s)_s + \alpha\big[Y(e^{2s} - e^{-2s})\big]_s = 0 \tag{3.5}$$

where we have replaced the variable x by s and introduced P by $P \equiv P(e^{-s}) = \alpha(e^{2s} + e^{-2s}) + \gamma$. To solve (3.5) we introduce an unknown function ϕ given by

$$Y = \phi_s \qquad \dot{\phi} = 2PY^2 - PY_s + \alpha(e^{2s} - e^{-2\alpha})Y \tag{3.6}$$

and reduce the problem to a linear equation for the function $\chi = e^{-2\phi}$,

$$P\chi_{ss} - A\chi_s + \chi_t = 0 \tag{3.7}$$

where $A = \alpha(e^{2s} - e^{-2s})$.

Shortly we will show that this equation is closely related to the Lame equation whose solutions can be given in terms of Jacobi elliptic functions. First, however, we would like to present some other forms of this equation which represent some other familiar problems.

To do this let us consider the stationary case, i.e. we assume that $\chi_t = \lambda\chi$ (bearing in mind that in the general case we can represent the t dependence of χ by $\int d\lambda\, e^{\lambda t}\chi(\lambda)\, d\lambda$). Then, it is possible to reduce (3.7) to a stationary one-dimensional Schrödinger equation $\psi'' + u\psi = 0$. To do this we eliminate the terms involving the first derivative by setting $\chi P^{-1/4} = \psi$ and find that the equation reduces to

$$\psi_{ss} + \left(\frac{1}{4} + \frac{(\lambda - \frac{3}{2}\gamma)}{P} + \frac{5}{4}\frac{g_1 g_2}{P^2}\right)\psi = 0 \tag{3.8}$$

where $g_{1,2} = \gamma \pm 2\alpha$.

In the cases of higher symmetry ($\alpha = 0$, or $\gamma = \pm 2\alpha$) (3.8) is solvable in terms of elementary functions. Note that if we rewrite (3.8) in terms of the original variable x (and not s) we find that the equation for the function $\psi = \chi/(ds/dx)$ (equation (3.7)) takes the form

$$\psi'' + \left(\lambda + 2\gamma - \frac{2g_1 g_2}{Q}\right)\psi = 0 \tag{3.9}$$

where $Q(x)$ may be expressed in terms of the Weierstrass doubly periodic $R0$ function, and in the case of the higher symmetry ($\alpha = 0$, or $g_{1,2} = \gamma \pm 2\alpha = 0$) (3.8) becomes the familiar heat equation.

Next we return to the stationary form of (3.7) and introduce $P = \alpha(e^{2s} + e^{-2s}) + \gamma$ as our independent variable. We obtain

$$\chi_{PP} + \frac{1}{2}\left(\frac{1}{P - g_1} + \frac{1}{P - g_2} - \frac{1}{P}\right)\chi_P + \frac{\lambda\chi}{4P(P - g_1)(P - g_2)} = 0\,. \tag{3.10}$$

This equation, Heun's equation, is very close to the Lame equation. To get the Lame equation we return to the stationary form of (3.7), differentiate it with respect to s and introduce a new function $\theta = \chi_s$. The equation for θ is already the Lame equation in the variable P, i.e.

$$\theta_{PP} + \frac{1}{2}\left\{\frac{1}{P - g_1} + \frac{1}{P - g_2} + \frac{1}{P}\right\}\theta_P = \frac{(\lambda + 2P - g_1 - g_2)}{4P(P - g_1)(P - g_2)}\theta\,. \tag{3.11}$$

In future investigations we will use solutions of this equation, which can be given in terms of elliptic functions, to obtain soliton solutions of the Landau–Lifschitz equation. In this work we restrict our attention to the general study of the problem and apply it to some simple cases.

4. Recurrence relations for the Landau–Lifschitz lattice

If we want to find a solution of the semi-infinite Landau–Lifschitz chain (starting from one end) then it is convenient to take the first step in the following form (bearing in mind that this form resembles the solution discussed in the previous section):

$$u_0 = \mathrm{e}^s \qquad v_0 = \mathrm{e}^{-s}\frac{1+(\ln\chi)'}{1-(\ln\chi)'} \tag{4.1}$$

where χ is an arbitrary function and, here and below, $'$ denotes $\mathrm{d}/\mathrm{d}s$. Then, let us seek solutions of (2.5) of the form

$$u_n = \mathrm{e}^s\frac{1-\sigma_{n-1}}{1+\sigma_{n-1}} \qquad v_n = \mathrm{e}^{-s}\frac{1+\sigma_n}{1-\sigma_n} \tag{4.2}$$

with the boundary conditions $\sigma_{-1} = 0$ and $\sigma_0 = (\ln\chi)'$.

In this notion the right-hand side of the chain equation (2.5) takes the form

$$\sigma_n - \frac{1-\sigma_n^2}{2}\frac{A_n}{B_n} \equiv \sigma_n - \frac{1-\sigma_n^2}{2}\delta_n \tag{4.3}$$

where

$$A_n = \left(P\left(\sigma_n' + \sigma_n^2\right) - A\sigma_n\right)'$$
$$B_n + \sigma_n A_n = \det\begin{pmatrix} P\sigma_n & \sigma_n'+\sigma_n^2-1 \\ P\sigma_n' - A\sigma_n & (\sigma_n'+\sigma_n^2-1)' \end{pmatrix}$$

and where the functions P and A are the same as in the previous section. The equation (2.5) itself becomes

$$\sigma_{n+1} = \frac{\delta_n - \frac{\sigma_{n-1}}{(1-\sigma_{n-1}\sigma_n)}}{1+\sigma_n\left[\frac{\delta_n-\sigma_{n-1}}{(1-\sigma_{n-1}\sigma_n)}\right]} \tag{4.4}$$

or, after some trivial manipulations,

$$\sigma_{n+1} = \frac{A_n - \sigma_{n-1}(B_n+\sigma_n A_n)}{(1-2\sigma_n\sigma_{n-1})(B_n+\sigma_n A_n)+\sigma_{n-1}\sigma_n^2 A_n}\,. \tag{4.5}$$

However, we can rewrite (4.4) in the form

$$2+\delta_n\sigma_n = \frac{1}{1-\sigma_n\sigma_{n+1}} + \frac{1}{1-\sigma_{n-1}\sigma_n} \tag{4.6}$$

or

$$\frac{1}{1-\sigma_n\sigma_{n+1}} = \sum_{\alpha=0}^{n}\sigma_{n-\alpha}\delta_{n-\alpha}(-1)^\alpha + 1\,. \tag{4.7}$$

Using these formulae we find that

$$\sigma_{-1} = 0 \qquad \sigma_0 = \frac{\chi'}{\chi} \qquad \sigma_1 = \frac{\det\begin{pmatrix} \chi & L \\ \chi' & L' \end{pmatrix}}{\det\begin{pmatrix} \chi' & \chi''-\chi \\ L & P(\chi'''-\chi') \end{pmatrix}} \tag{4.8}$$

where $L \equiv \hat{L}(\chi) = P\chi'' - A\chi'$.

All calculations can be carried out all the way to the end so that we can obtain an expression for σ_{n+1} in the form of a ratio of two determinants of the $(n+2)$ order. The calculations are simpler in the case of axial symmetry (when, for instance, $\alpha = 0$). This special case is of interest by itself and in physical applications it is referred to as the *XXY* model or the model of the uniaxial anisotropic system. We will discuss this case in the next section, where we present an explicit calculation of σ_2 and σ_3 in the case when $\alpha = 0$. In the appendix we generalize these results to arbitrary σ_n. These calculations will serve us as a 'warm-up exercise' for the general case which will be discussed in following sections.

5. Calculation of σ_2 and σ_3 when $\alpha = 0$

First we introduce the following abbreviations. The (arbitrary) function χ will be denoted by 0 (thus $0 = \chi$) and its pth derivative will be denoted by p (thus $p = \mathrm{d}^p\chi/\mathrm{d}s^p$). Moreover, in the next section we shall use the symbol $\widehat{p}$ for the expression $\widehat{p} \equiv p - (p-2) = (\mathrm{d}^p\chi/\mathrm{d}s^p) - (\mathrm{d}^{p-2}\chi/\mathrm{d}s^{p-2})$, which, as we will see, often arises in our calculations. Note that $\hat{s}' = (s+1) - (s-1) = \widehat{(s+1)}$. In this section we show that, when $\alpha = 0$, σ_2 and σ_3 are given by

$$\sigma_2 = \frac{\det\begin{pmatrix} 1 & 20 & 31 \\ 2 & 31 & 42 \\ 3 & 42 & 53 \end{pmatrix}}{\det\begin{pmatrix} 0 & 20 & 31 \\ 1 & 31 & 42 \\ 2 & 42 & 53 \end{pmatrix}} \qquad \sigma_3 = \frac{\det\begin{pmatrix} 0 & 20 & 31 & 42 \\ 1 & 31 & 42 & 53 \\ 2 & 42 & 53 & 64 \\ 3 & 42 & 53 & 64 \end{pmatrix}}{\det\begin{pmatrix} 1 & 20 & 31 & 42 \\ 2 & 31 & 42 & 53 \\ 3 & 42 & 53 & 64 \\ 4 & 53 & 64 & 75 \end{pmatrix}}. \tag{5.1}$$

From these expressions it is easy to guess the form of a general σ_k.

To prove (5.1) we observe that (4.4) tells us that σ_2 is given by

$$\sigma_2 = \frac{\ln'\left[\sigma_1' + (\sigma_1)^2 - 1\right](1 - \sigma_1\sigma_0) + \sigma_1'\sigma_0}{\sigma_1\left\{(1 - \sigma_1\sigma_0)\left[\ln\left(\left(\sigma_1' + (\sigma_1)^2 - 1\right)/\sigma_1\right)\right]' + \sigma_1'\sigma_0\right\}} \tag{5.2}$$

and a similar expression holds for σ_3 (the indices on all σ's have to be increased by 1). So we see that to calculate σ_2 (and σ_3) we need σ_1' (σ_2'), $\sigma_1' + (\sigma_1)^2 - 1$ ($\sigma_2' + (\sigma_2)^2 - 1$), $1 - \sigma_0\sigma_1$ ($1 - \sigma_1\sigma_2$) and $\sigma_1'\sigma_0$ ($\sigma_2'\sigma_1$).

Let us first calculate σ_1' (σ_2'). To do this we observe that if we put $\sigma_i = s_1/s_2$ then

$$\sigma_i = \frac{s_1's_2 - s_1s_2'}{(s_2)^2}. \tag{5.3}$$

To perform the differentiation of our determinants we observe that our matrices have the property that each of their rows is the derivative of the row above. Hence, when we differentiate their determinants it is sufficient to differentiate only the last row, as all the other contributions vanish. The differentiation of the last row increases all the integers in it by one (i.e. 2 becomes 3, 42 becomes 53, etc) and so we find that the numerator of (5.3) (in the σ_2 case) becomes

$$\det\begin{pmatrix}1 & 20 & 31\\ 2 & 31 & 42\\ 4 & 53 & 64\end{pmatrix}\det\begin{pmatrix}0 & 20 & 31\\ 1 & 31 & 42\\ 2 & 42 & 53\end{pmatrix}-\det\begin{pmatrix}1 & 20 & 31\\ 2 & 31 & 42\\ 3 & 42 & 53\end{pmatrix}\det\begin{pmatrix}0 & 20 & 31\\ 1 & 31 & 42\\ 3 & 53 & 64\end{pmatrix}$$
$$=\det\begin{pmatrix}20 & 31\\ 31 & 42\end{pmatrix}\det\begin{pmatrix}0 & 1 & 2 & 3\\ 1 & 2 & 3 & 4\\ 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 6\end{pmatrix}. \tag{5.4}$$

This result follows from the Jacobi identity for the determinants [8] (the determinants of the matrices, which have $\begin{pmatrix}20 & 31\\ 31 & 42\end{pmatrix}$ in common). Similarily we find that

$$\sigma_1' = -\frac{20\det\begin{pmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 4\end{pmatrix}}{\left(\det\begin{pmatrix}1 & 20\\ 2 & 31\end{pmatrix}\right)^2}. \tag{5.5}$$

Next we calculate

$$\sigma_i' + (\sigma_i)^2 - 1 = \frac{s_2(s_1' - s_2) - s_1(s_2' - s_1)}{s_2^2}. \tag{5.6}$$

This time we have to calculate $s_2' - s_1$ and $s_1' - s_2$. We expand each expression along the last row and find that, say, in the $i = 2$ case

$$s_1' - s_2 = \det\begin{pmatrix}1 & 20 & 42\\ 2 & 31 & 53\\ 3 & 42 & 64\end{pmatrix} \qquad s_2' - s_1 = \det\begin{pmatrix}0 & 20 & 42\\ 1 & 31 & 53\\ 2 & 42 & 64\end{pmatrix} \tag{5.7}$$

and so

$$\begin{aligned}\sigma_2' + (\sigma_2)^2 - 1 &= \frac{\det\begin{pmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 4\end{pmatrix}\det\begin{pmatrix}20 & 31 & 42\\ 31 & 42 & 53\\ 42 & 53 & 64\end{pmatrix}}{\left(\det\begin{pmatrix}0 & 20 & 31\\ 1 & 31 & 42\\ 2 & 42 & 53\end{pmatrix}\right)^2} \equiv \frac{\alpha\beta}{\gamma^2}\\ \sigma_1' + \sigma_1^2 - 1 &= -\frac{\det\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\det\begin{pmatrix}20 & 31\\ 31 & 42\end{pmatrix}}{\left(\det\begin{pmatrix}1 & 20\\ 2 & 31\end{pmatrix}\right)^2}.\end{aligned} \tag{5.8}$$

In a similar way we find that

$$1-\sigma_1\sigma_2=\frac{\det\begin{pmatrix}0&1&2\\1&2&3\\2&3&4\end{pmatrix}\det\begin{pmatrix}20&31\\31&42\end{pmatrix}}{\det\begin{pmatrix}1&20\\2&31\end{pmatrix}\det\begin{pmatrix}0&20&31\\1&31&42\\2&42&53\end{pmatrix}}\tag{5.9}$$

with a similar expression for $1-\sigma_1\sigma_0$.

Next we calculate the derivative of $\log(\alpha\beta/\gamma^2)$, which appears in (5.2), where α, β and γ are defined as in (5.8). This means that we have to calculate $(\beta\gamma\alpha'+\beta'\alpha\gamma-2\gamma'\alpha\beta)/\alpha\beta\gamma$

Putting in the concrete expressions for α, β and γ in the σ_2 case we find that

$$\alpha'\gamma-\gamma\alpha'=-\det\begin{pmatrix}0&20\\1&31\end{pmatrix}\det\begin{pmatrix}0&1&2&3\\1&2&3&4\\2&3&4&5\\3&4&5&6\end{pmatrix}\tag{5.10}$$

and

$$\beta'\gamma-\gamma'\beta=\det\begin{pmatrix}20&31\\31&42\end{pmatrix}\det\begin{pmatrix}0&20&31&42\\1&31&42&53\\2&42&53&64\\3&53&64&75\end{pmatrix}.\tag{5.11}$$

Finally, collecting all the terms, factoring them out etc, we find that σ_3 is given by a ratio of two expressions, the numerator of which involves a product of two determinants and the denominator a difference of two products. The denominator is given by

$$\det\begin{pmatrix}0&20&31&42\\1&31&42&53\\2&42&53&64\\3&53&64&75\end{pmatrix}\det\begin{pmatrix}1&20&31\\2&31&42\\3&42&53\end{pmatrix}-\det\begin{pmatrix}0&1&2&3\\1&2&3&4\\2&3&4&5\\3&4&5&6\end{pmatrix}\det\begin{pmatrix}20&31&42\\31&42&53\\42&53&64\end{pmatrix}\tag{5.12}$$

which we calculate by expanding along the last row. We find that this expression is given by

$$\det\begin{pmatrix}0&20&31\\1&31&42\\2&42&53\end{pmatrix}\det\begin{pmatrix}1&20&31&42\\2&31&42&53\\3&42&53&64\\4&53&64&75\end{pmatrix}.\tag{5.13}$$

This is all that is required to demonstrate our claim that σ_3 (and σ_2) are given by (5.1).

6. An example

In this section we determine explicitly some solutions of (2.3) in the case when $\alpha=0$. We leave the discussion of the generality of these solutions to a further publication. Here we just want to show that our procedure is sound, i.e. that we can find non-trivial solutions of the Landau–Lifschitz equation with all the reality conditions properly imposed. So, here, we look for simple non-trivial solutions of (2.3).

To find such solutions we have to first solve (3.7) for the function χ. However, for $\alpha = 0$ this equation is just the heat equation and it is easy to write down its solutions. The simplest solution (apart from $\chi = 0$) is probably

$$\chi(x,t) = A\exp(\mathrm{i}\theta) \tag{6.1}$$

where $\theta = \mu^2 t + \mu x^2$, and where μ and A are arbitrary complex numbers. Here, for simplicity, we have chosen, $\gamma = 1$ in the definition of $P(y)$ in (2.3), which gave us $s = \mathrm{i}x$.

With this choice we find that $\sigma_0 =$ constant and so we see that we have obtained a time-independent solution for the Landau–Lifschitz field u. To find a time-dependent field we need to start with a more complicated χ.

To do this we consider a slightly more general solution of (3.7); namely

$$\chi = \sum_{k=0}^{3} A_k \exp(\mathrm{i}\theta_k) \tag{6.2}$$

where $\theta_k = \mu_k^2 t + \mu_k x^2$, and where A_k and μ_k are, at this stage, some arbitrary complex numbers. With this choice of χ we find that

$$\sigma_0 = \frac{\sum_{k=0}^{3} A_k\mu_k \exp(\mathrm{i}\theta_k)}{\sum_{k=0}^{3} A_k \exp(\mathrm{i}\theta_k)} \tag{6.3}$$

and

$$\sigma_1 = \frac{\sum_{k=0}^{3} (1/A_k)(\mu_i - \mu_j)^2(\mu_i + \mu_j)\exp(-\mathrm{i}\theta_k)}{\sum_{k=0}^{3} (1/A_k)(\mu_i - \mu_j)^2(1 + \mu_i\mu_j)\exp(-\mathrm{i}\theta_k)} \qquad k \neq \mathrm{i} \neq j\,. \tag{6.4}$$

Next we impose our conditions of reality of u. To do this we require that $v_2^\dagger = u_0$. This will be true if $\sigma_2 = 0$. When we calculate σ_2 we find that

$$\sigma_2 = \frac{\mu_1 + \mu_2 + \mu_3 + \mu_1\mu_2\mu_3}{1 + \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3} \tag{6.5}$$

and so we see that we have a condition on μ's:

$$\mu_1 + \mu_2 + \mu_3 + \mu_1\mu_2\mu_3 = 0\,. \tag{6.6}$$

However, this is not enough; to guarantee the reality of our solution of the Landau–Lifschitz equation we also have to impose $u_2^\dagger = v_0$. Given equation (4.2) we see that this requires $\sigma_0^\dagger = -\sigma_1$. In this case our solution is given by $u = u_1$.

It is easy to see that there exist many solutions of (6.6) and of $\sigma_0^\dagger = -\sigma_1$. To see this we expand all the terms in $\sigma_0^\dagger = -\sigma_1$ and find that many of the resultant conditions are satisfied if μ_k satisfy (6.6) and either all are real (i.e. $\mu_k = \mu_k^\dagger$) or one μ is real and the remaining two are complex conjugates of each other.

The remaining conditions can then be treated as conditions on A_k. In the first case (when all μ_k are real) A_k have to satisfy

$$
\begin{aligned}
&\frac{A_1^\dagger}{A_1}(\mu_1-\mu_3)^2(\mu_1+\mu_3)\left(1-\frac{\mu_1}{\mu_2}\right)+\frac{A_2^\dagger}{A_1}(\mu_2-\mu_3)^2(\mu_2+\mu_3)\left(1-\frac{\mu_2}{\mu_1}\right)=0\\
&\frac{A_1^\dagger}{A_3}(\mu_1-\mu_2)^2(\mu_1+\mu_2)\left(1-\frac{\mu_1}{\mu_3}\right)+\frac{A_3^\dagger}{A_1}(\mu_2-\mu_3)^2(\mu_2+\mu_3)\left(1-\frac{\mu_3}{\mu_1}\right)=0\\
&\frac{A_2^\dagger}{A_3}(\mu_1-\mu_2)^2(\mu_1+\mu_2)\left(1-\frac{\mu_2}{\mu_3}\right)+\frac{A_3^\dagger}{A_2}(\mu_1-\mu_3)^2(\mu_1+\mu_3)\left(1-\frac{\mu_3}{\mu_2}\right)=0
\end{aligned}
\tag{6.7}
$$

while in the second case (when $\mu_2=\mu_2^\dagger$, $\mu_1=\mu_3^\dagger$) we have

$$
\begin{aligned}
\frac{A_1^\dagger}{A_1}(\mu_2-\mu_3)^2(1+\mu_2\mu_3)&=\frac{A_3^\dagger}{A_3}(\mu_1-\mu_2)^2(1+\mu_1\mu_2)\\
\frac{A_1^\dagger}{A_2}(\mu_1-\mu_3)^2(1+\mu_1\mu_3)&=\frac{A_2^\dagger}{A_3}(\mu_1-\mu_2)^2(1+\mu_1\mu_2)\\
\frac{A_2^\dagger}{A_1}(\mu_2-\mu_3)^2(1+\mu_2\mu_3)&=\frac{A_3^\dagger}{A_2}(\mu_1-\mu_3)^2(1+\mu_1\mu_3)\,.
\end{aligned}
\tag{6.8}
$$

Notice that we can set, say, $A_1=1$ and that, say, the last equation in (6.7) and in (6.8) is automatically fulfilled if all the other equations are satisfied.

There are various solutions of these equations. In the case when μ_k are real a natural choice is to take A_k real. Then (6.7) become equations for A_k^2, which should have positive solutions. Thus we have to find μ_k, $k=1$, 2 and 3 such that (6.6) is satisfied and A_k^2 are positive. It is easy to check that if we choose, say, $\mu_1=0.5$, $\mu_2=0.1$ (and μ_3 is given by (6.6)) the expressions for A_k^2 are indeed positive and we have an explicit solution of the Landau–Lifschitz equation.

In the second case the situation is even simpler, as this time (6.8) are satisfied if we take, say, $A_2=1$ and

$$
A_1=\frac{1}{A_3^\dagger}\frac{(\mu_2-\mu_3)^2}{(\mu_1-\mu_3)^2}\frac{(1+\mu_2\mu_3)}{(1+\mu_1\mu_3)}\,.
\tag{6.9}
$$

In this case it is easy to see that the solution u is of the 'soliton'-type—i.e. S_3 (see equation (2.2)) is non-zero in a localized region.

We will not discuss any further properties of these solutions, nor try to construct more general ones. This will be done in our further work. Our reason for giving these examples here has been to demonstrate that the method works, i.e. that it gives us solutions of the original equation with all the reality conditions explicitly fulfilled.

7. The general case ($\alpha \neq 0$)

In this section we will present explicit formulae for the recurrence relation for σ_k (4.5) in the general case (i.e. with no symmetry). We do not give a proof of the derived expression as our method of proving it is quite involved and, in the main, follows quite closely the steps used in the proofs given in the previous sections, except that all the calculations are much more cumbersome. We believe that a simpler proof of our results may be found by considering, in more detail, the group-theoretic nature of the Landau–Lifschitz chain with two fixed ends (i.e. limited from both ends) and given in terms of the theory of semi-simple algebras and their representations, but so far we have not found it yet.

To present the explicit expression for σ_k in the general case we introduce the following infinite-dimensional matrix:

$$\begin{pmatrix} & \chi' & \chi''-\chi & L' & L''-L & (L^2)' & \dots \\ \chi & L & R(\chi'''-\chi') & L^2 & R(L'''-L') & L^3 & \dots \\ \chi' & L' & L''-L & (L^2)' & L^{iv}-L'' & (L^3)' & \dots \\ L & L^2 & R(L'''-L') & (L^3) & R(L^{v}-L''') & L^4 & \dots \\ L' & (L^2)' & R(L^{iv}-L'') & (L^3)' & L^{vi}-L^{iv} & (L^4)' & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} . \tag{7.1}$$

The form of this matrix is, hopefully, obvious from the entries that are given explicitly in (7.1). Then, our expression for σ_k is derived from (7.1) and is given by

$$\sigma_k = \left(\frac{\det_{k+1}||\chi'||}{\det_{k+1}||\chi||}\right)^{\epsilon_k} \qquad k \geqslant 0 \tag{7.2}$$

where the symbol $||\chi||$ denotes the matrix of the $(k+1)$th order which is obtained from (7.1) by taking its submatrix consisting of entries only in rows $2, 3, \dots k+2$ and columns $1, 2, \dots k+1$, and $||\chi'||$ denotes a similar matrix with entries from rows $1, 2, \dots k+1$ and columns $2, 3, \dots k+2$. Also $\epsilon_k = (-1)^k$.

8. Conclusions

We have shown in this paper how the exploitation of the existence of a discrete group of symmetries of a given equation can help us in determining its solutions. In this paper we have demonstrated this for the example of the Heisenberg ferromagnet (with uniaxial or biaxial anisotropy) described by the Landau–Lifschitz equations. Our main result for these equations is given by (7.1).

We would like to stress that, although in this paper we have concentrated our attention on the Landau-Lifschitz equations, our method is completely general and could be applied to any completely integrable systems. In each of these cases one has to take the following steps: first, one has to rewrite the equations which describe the group of the discrete symmetries as a system of equations for the corresponding lattice. This system of equations is completely integrable due to the integrability of of the symmetry equations. Next one derives the solutions of the lattice equations which arise when we require that this chain of equations is limited from both ends. The solutions one obtains depend on several arbitrary complex parameters. Finally, we impose the condition of reality (this we do by requiring that $v_0 = u_N^\dagger$ and $u_0 = v_N^\dagger$ (N-even), where we have assumed that the finite chain stretches from 0 to N. This condition of reality fixes some of our free parameters. Then, the solution in the middle of the chain satisfies the reality condition which we want our solution to possess, namely $v_{N/2} = u_{N/2}^\dagger$.

Appendix

Here we generalize the results of section 5 to the case of arbitrary n. We will use the induction method to prove our general formulae for σ_k ($k \geqslant 0$).

In our general proof we will use the notation of section 5, i.e. we put 0 for χ and p for and its pth derivative. The matrices which arise in our calculation have the property that all their rows can be obtained from a row of functions by differentiating them a certain number of times. So, we will denote them by $||a_1, a_2, \dots, a_n; s_1, s_2, \dots, s_n||$ with the understanding that the lth row of such a matrix is given by $a_1^{s_l}, a_2^{s_l}, \dots, a_n^{s_l}$, where a_i are some functions of s and s_i some non-negative integers.

The determinants of such matrices will be denoted by

$$\det ||a_1, a_2, \dots, a_n; s_1, s_2, \dots, s_n|| = (a_1, a_2, \dots, a_n; s_1, s_2, \dots, s_n)\,. \tag{A.1}$$

In this notation, the general form of the solution of the Landau–Lifschitz chain is given by

$$\begin{aligned} \sigma_{2n} &= \frac{(1, \widehat{2}, \widehat{3}, \dots, \widehat{2n+1}; 0, 1, \dots, 2n)}{(0, \widehat{2}, \dots, \widehat{2n+1}; 0, 1, \dots, 2n)} = \frac{s^1_{2n}}{s^2_{2n}} \\ \sigma_{2n+1} &= \frac{(0, \widehat{2}, \widehat{3}, \dots, \widehat{2n+2}; 0, 1, \dots, 2n+1)}{(1, \widehat{2}, \widehat{3}, \dots, \widehat{2n+2}; 0, 1, \dots, 2n+1)} = \frac{s^1_{2n+1}}{s^2_{2n+1}} \end{aligned} \tag{A.2}$$

where $n \geqslant 0$, $\widehat{1} = 1$, and $(-1) \equiv 0$.

To prove this we observe that our general formulae for σ_{i+1} (4.5) involve σ_i', $\sigma_i' + \sigma_i^2 - 1$ and $1 - \sigma_{i-1}\sigma_i$. We shall calculate them in an explicit form; afterwards, the checking of the final expression reduces to a very simple problem.

For definiteness we shall limit ourselves to the case of odd i (the case of even i is very similar). First we calculate σ'_{2n+1}. We find

$$\begin{aligned} &\frac{(0, \widehat{2}, \dots, \widehat{2n+2}; 0, 1, \dots, 2n, 2n+2)(1, \widehat{2}, \dots, \widehat{2n+2}; 0, 1, \dots, 2n+1)}{(s^2_{2n+1})^2} \\ &\qquad - \frac{(0, \widehat{2}, \widehat{3}, \dots, \widehat{2n+2}; 0, 1, \dots, 2n+1)(1, \widehat{2}, \dots, \widehat{2n+2}; 0, 1, \dots, 2n, 2n+2)}{(s^2_{2n+1})^2} \\ &= \frac{(0, 1, \dots, 2n+2; 0, 1, \dots, 2n+2)(\widehat{2}, \widehat{3}, \dots, \widehat{2n+2}; 0, 1, \dots, 2n)}{(s^2_{2n+1})^2}\,. \end{aligned} \tag{A.3}$$

The last equality, again, follows from the Jacobi identity [8].

To calculate $\sigma_i' + \sigma_i^2 - 1$ and $1 - \sigma_{i-1}\sigma_i$ we use the following property of determinants (which can be checked by a direct calculation):

$$\begin{aligned} &\det ||l_1; a_1, \dots, a_{n-1}; b_1|| \det ||l_2; a_1, \dots, a_{n-1}; b_2|| \\ &\qquad - \det ||l_2; a_1, \dots, a_{n-1}; b_1|| \det ||l_1; a_1, \dots, a_{n-1}; b_2|| \\ &\quad = \det ||l_1, l_2; a_1, \dots, a_{n-1}|| \det ||a_1, \dots, a_{n-1}, b_1, b_2|| \end{aligned} \tag{A.4}$$

where a_i, b_1, b_2, l_1 and l_2 denote arbitrary $(n+1)$-dimensional column vectors.

Next, using the notation of (A.2), we observe that, as in (5.6),

$$\sigma_i' + \sigma_i^2 - 1 = \frac{s_i^2((s_i^1)' - s_i^2) - s_i^1((s_i^2)' - s_i^1)}{(s_i^2)^2}. \tag{A.5}$$

We calculate the differences of the terms in the numerator of (A.5). Thus for $(s_{2n+1}^1)' - s_{2n+1}^2$ we have

$$\begin{aligned}
(s_{2n+1}^1)' - s_{2n+1}^2 &= (0, \widehat{2}, \widehat{3}, \ldots, \widehat{2n+2}; 0, 1, \ldots, 2n, 2n+2) \\
&\quad - (1, \widehat{2}, \ldots, \widehat{2n+2}; 0, 1, \ldots, 2n+1) \\
&= 0(\widehat{2}, \widehat{3}, \ldots, \widehat{2n+2}; 1, \ldots, 2n+2n+2) \\
&\quad - 1\big[(\widehat{2}, \ldots, \widehat{2n+2}; 0, 2, \ldots, 2n, 2n+2) + (\widehat{2}, \ldots, \widehat{2n+2}; 1, \ldots, 2n+1)\big] \\
&\quad + 2\big[(\widehat{2}, \ldots, \widehat{2n+2}; 0, 1, 3, \ldots, 2n, 2n+2) \\
&\quad + (\widehat{2}, \ldots, \widehat{2n+2}; 0, 2, \ldots, 2n+1)\big] \\
&\quad + (-1)^k\big[(\widehat{2}, \ldots, \widehat{2n+2}; 0, \ldots, k-1, k+1, \ldots, 2n, 2n+2) \\
&\quad + (\widehat{2}, \ldots, \widehat{2n+2}; 0, \ldots, k-2, k, \ldots, 2n+1)\big] + \cdots.
\end{aligned} \tag{A.6}$$

The simplest way to calculate the sums of determinants in square brackets in (A.6) is to represent the functions $\widehat{m}$ by their Laplace transforms, i.e. put $\widehat{2} = \int e^{\lambda s}\phi(\lambda)\,d\lambda$, $\widehat{3} = \int \lambda e^{\lambda s}\phi(\lambda)\,d\lambda$, etc. Then each of the determinants may be represented by an $(2n+1)$-dimensional integral over $d\lambda_1 \ldots d\lambda_{2n+1}$ with the integrand being proportional to the Vandermonde determinant $W(\lambda_1, \ldots \lambda_{2n+1})$, which is antisymmetric with respect to the permutation of any pair of its arguments.

Let us calculate, for example, the sum of the two determinants which multiplies (1) in (A.6). We find

$$\begin{aligned}
&\int d\lambda_1 \ldots d\lambda_{2n+1}\, \phi(\lambda_1)\phi(\lambda_2)\ldots\phi(\lambda_{2n+1}) \\
&\qquad \times \big[\lambda_2^2\lambda_3^3\ldots\lambda_{2n+1}^{2n+2} + \lambda_1\lambda_2^2\ldots\lambda_{2n+1}^{2n+1}\big] W(\lambda_1\ldots\lambda_{2n+1}) \\
&\quad = \int d\lambda_1 \ldots d\lambda_{2n+1}\, \phi(\lambda_1)\ldots\phi(\lambda_{2n+1}) \\
&\qquad \times \lambda_2^2\lambda_3^3\ldots\lambda_{2n+1}^{2n+1}\left(\lambda_1 + \lambda_{2n+1} + \sum_{i=2}^{2n}\lambda_i\right) W(\lambda_1\ldots\lambda_{2n+1})
\end{aligned} \tag{A.7}$$

as each of the terms of the added sum corresponds to an expression which is symmetric with respect to the permutation of any two indices of λ's and so vanishes after the integration with the Vandermonde determinant.

But the last step of (A.7) can be rewritten as

$$\begin{aligned}
&\int d\lambda_1 \ldots d\lambda_{2n+1}\, \phi(\lambda_1)\ldots\phi(\lambda_{2n+1})\lambda_2^2\lambda_3^3\ldots\lambda_{2n+1}^{2n+1}\begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{2n} & \lambda_1^{2n+2} \\ 2 & \lambda_2 & \ldots & \lambda_2^{2n} & \lambda_2^{2n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_{2n+1} & \ldots & \lambda_{2n+1}^{2n} & \lambda_{2n+1}^{2n+2} \end{pmatrix} \\
&\quad = (\widehat{2}, \widehat{3}, \ldots, \widehat{2n+1}, \widehat{2n+3}; 0, 1, 2, \ldots, 2n+1).
\end{aligned} \tag{A.8}$$

Performing such a calculation for each term in (A.6) and in (A.5) allows us to prove our formula (A.2).

In the same way we calculate all the other terms in (A.6) and find

$$\begin{aligned}(s^1_{2n+1})' - s^2_{2n+1} &= (0, \widehat{2}, \widehat{3}, \ldots, \widehat{2n+2}; 0, 1, \ldots, 2n, 2n+2) \\ &\quad - (\widehat{1}, \widehat{2}, \ldots, \widehat{2n+2}; 0, 1, \ldots, 2n+1) \\ &= (0, \widehat{2}, \widehat{3}, \ldots, \widehat{2n+1}, \widehat{2n+3}; 0, 1, \ldots, 2n, 2n+1)\,. \end{aligned} \tag{A.9}$$

The calculations of the second term in the numerator are the same and we obtain

$$(s^2_{2n+1})' - s^1_{2n+1} = (1, \widehat{2}, \widehat{3}, \ldots, \widehat{2n+1}, \widehat{2n+3}, 0, 1, \ldots, 2n, 2n+1)\,. \tag{A.10}$$

In this way we find that

$$\sigma'_{2n+1} + \sigma^2_{2n+1} - 1 = \frac{(0, 1, 2, \ldots, 2n+1; 0, 1, \ldots, 2n+1)(\widehat{2}, \widehat{3}, \ldots, \widehat{2n+2}, \widehat{2n+3}; 0, 1, \ldots, 2n+1)}{s^2_{2n+1}}\,. \tag{A.11}$$

The calculations of $\sigma'_i + \sigma^2_i - 1$ in the case of even i are very similar. In the case when $i = 2n$ we find

$$\sigma'_{2n} + \sigma^2_{2n} - 1 = \frac{(0, 1, \ldots, 2n, 0, 1, \ldots, 2n)(\widehat{2}, \widehat{3}, \ldots, \widehat{2n+2}; 0, 1, \ldots, 2n)}{s^2_{2n}}\,. \tag{A.12}$$

When we calculate $1 - \sigma_{i-1}\sigma_i$ it is necessary to use the definition (A.2), expand the determinants of the ith order along the last row, use (A.4) and then observe that

$$1 - \sigma_{i-1}\sigma_i = \frac{(0, 1, \ldots, i; 0, 1, \ldots i)(\widehat{2}, \widehat{3}, \ldots \widehat{i+1}; 0, 1, \ldots, i-1)}{(1, \widehat{2}, \ldots, \widehat{i}; 0, 1, \ldots, i-1)(0, \widehat{2}, \ldots, \widehat{i+1}; 0, \widehat{2}, \ldots, \widehat{i+1}, 0, 1, \ldots, i)}\,. \tag{A.13}$$

Finally, repeating, step by step, all the calculations (5.10)–(5.13) of the previous section we have convinced ourselves about the validity of our recurrant formulae (A.2).

Let us add that as a result of our formulae we can make the following statement: A general solution of the following one-dimensional chain system,

$$\ddot{\varphi}_i = \left(\frac{1}{e^{\varphi_{i+1}-\varphi_i} + 1} - \frac{1}{e^{\varphi_i - \varphi_{i-1}} + 1}\right)\dot{\varphi}_i(\dot{\varphi}_i + 2\gamma) \tag{A.14}$$

where γ is an arbitrary constant, $\dot{\varphi} = \partial\varphi/\partial t$ and on which the following boundary conditions are imposed $\varphi_{-1} = \varphi_{n+1} = 0$, is given by the expressions

$$e^{-\varphi_i} = \frac{1 + \sigma_i}{1 - \sigma_i}\,. \tag{A.15}$$

Here σ_i are defined by (A.2), χ are given by $\chi = \sum_{s=0}^{n} C_s e^{\lambda_s t}$, where C_s are arbitrary constants, and where λ_s satisfy the following relation:

$$\sum_\alpha \lambda_\alpha + \sum_{\alpha\neq\beta\neq\gamma} \lambda_\alpha\lambda_\beta\lambda_\gamma + \sum_{\substack{\text{odd number of indices,}\\ \text{all different}}} \lambda_\alpha \ldots \lambda_\gamma = 0\,. \tag{A.16}$$

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